How Undergraduates Compute Pure Nash Equilibria in Strategic Games

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Abstract

In 2x2 games we often solve them by writing arrows between outcomes in the matrix. I make this procedure precise and generalize it to all strategic games using some very basic category theory. I define a strategic game in categorical form in which outcomes are objects and preferences are arrows. A subcategory is formed by exclusion of arrows between outcomes that can not be reached unilaterally. I show that Nash equilibria in pure strategies are never in the domain of non-isomorphisms of the subcategory, i.e. a Nash equilibrium is in the domain of an isomorphism and not in the domain of an arrow otherwise. If additionally the preference relations are strict then the terminal object (if it exists) of this subcategory is equivalent to the unique Nash equilibrium in pure strategies. I use the homesets of this subcategory to obtain a simple function that attains its maximum in Nash equilbrium.

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1 Introduction

Strategic games in matrix form are very popular in economics, probably because it their simplicity. In undergraduate studies we learn how to solve for Nash equilibria in pure strategies in bi-matrix games by the means of drawing arrows in the 2x2 matrices or marking the most preferred outcome(s) for each player in large matrices. E.g. given row player's action, what is (are) the column player's most preferred action(s). The following picture illustrates the undergraduate Nash procedure in a 2x2 game:

$$\begin{array}{cccc} & L & & R \\ T & (4,4) & \longleftarrow & (0,2) \\ & \Big\uparrow & & \Big\downarrow \\ D & (2,0) & \longrightarrow & (3,3) \end{array}$$

Note that the horizontal arrows only concern the preferences of the column player, whereas the vertical arrows concern the row player.

Can this procedure made precise and generalized? A collection of outcomes and arrows between them can be interpreted as a mathematical category. In this paper I will only require some basic notions of category theory introduced in the next section (see [1] Mac Lane, 1971 for more material) that will serve as a formal framework of my investigation. In the third section I introduce a strategic game in categorical form and generalize the "undergraduate Nash procedure". Moreover, I obtain a simple function that attains its maximum in Nash equilibrium in pure strategies.

2 Some Basic Notions of Category Theory

Definition 1 (Category) *A category $\mathcal{C} = \langle \mathcal{C}_0, \mathcal{C}_1 \rangle$ is a collection $\mathcal{C}_0$ of objects and a collection $\mathcal{C}_1$ of morphisms which satisfy following structure:*

(i) Each morphism f has a domain X (or dom(f)) and codomain Y (or cod(f)) which are objects, written $f : X \longrightarrow Y$ or $X \xrightarrow{f} Y$.

(ii) Given two morphisms f and g such that cod(f) = dom(g), the composition of f and g, written $g \circ f$, is defined and has domain dom(f) and codomain cod(g): $g \circ f : X \longrightarrow Z$ or

$$X \xrightarrow{f} Y \xrightarrow{g} Z.$$

(iii) *Composition is associative, that is given* $f : X \longrightarrow Y$*,* $g : Y \longrightarrow Z$ *and* $h : Z \longrightarrow W$*,* $h \circ (g \circ f) = (h \circ g) \circ f$*.*

(iv) *For every object* X *there is an identity morphism* id_X*, satisfying* $id_X \circ g = g$ *for every* $g : Y \longrightarrow X$ *and* $f \circ id_X = f$ *for every* $f : X \longrightarrow Y$*.*

Note that a category may be empty. In some texts, morphisms are referred to as "arrows" and source and target.

Definition 2 (Homset) *If a collection of all morphisms* f *with* $dom(f) = X$ *and* $cod(f) = Y$ *is a set then it is denoted by* $Hom(X, Y)$ *called homset. In this case the category is said locally small. The category is small if all collections of morphisms in the category form homsets.*

Definition 3 (Subcategory) *A subcategory* $\mathcal{D}$ *of a category* $\mathcal{C}$ *is a pair of subsets* $\mathcal{D}_0$ *and* $\mathcal{D}_1$ *of objects and morphisms respectively s.t.*

(i) *if* $f \in \mathcal{D}_1$*, then* $dom(f)$*,* $cod(f) \in \mathcal{D}_0$*,*

(ii) *if* $C \in \mathcal{D}_0$*, then* $id_C \in \mathcal{D}_1$*,*

(iii) *if* $f, g \in \mathcal{D}_1$ *are a composable pair of morphisms then* $g \circ f \in \mathcal{D}_1$*.*

A subcategory is full if for any $C, D \in \mathcal{D}_0$*, if* $f : C \longrightarrow D$ *in* $\mathcal{C}$*, then* $f \in \mathcal{D}_1$*.*

Definition 4 (Isomorphism) *An morphism* $f : A \longrightarrow B$ *in a category* $\mathcal{C}$ *is an isomorphism if it has an inverse, i.e. a morphism* $g : B \longrightarrow A$ *for which* $f \circ g = id_B$ *and* $g \circ f = id_A$*.*

Definition 5 (Terminal Object) *An object* T *of a category* $\mathcal{C}$ *is called terminal if there is exactly one morphism* $A \longrightarrow T$ *for each object of* $\mathcal{C}$*.*

Definition 6 (Functor) *Given two categories* $\mathcal{C}$ *and* $\mathcal{D}$*, a functor* $F : \mathcal{C} \longrightarrow \mathcal{D}$ *consists of operations* $F_0 : \mathcal{C}_0 \longrightarrow \mathcal{D}_0$ *and* $F_1 : \mathcal{C}_1 \longrightarrow \mathcal{D}_1$ *such that for each* $f : X \longrightarrow Y$*,* $F_1(f) : F_0(X) \longrightarrow F_0(Y)$ *and*

(i) *for* $X \xrightarrow{f} Y \xrightarrow{g} Z$*,* $F_1(g \circ f) = F_1(g) \circ F_1(f)$*;*

(ii) $F_1(id_X) = id_{F_0(X)}$ *for each* $X \in \mathcal{C}_0$*.*

Definition 7 (Fully Faithful) *A functor* $F : \mathcal{C} \longrightarrow \mathcal{D}$ *is called faithful if it is injective when restricted to each homset (one-to-one).*[1] *A functor* $F : \mathcal{C} \longrightarrow \mathcal{D}$ *is called full if it is surjective*[2] *on each homset, i.e. if for every two objects* A, B *of* $\mathcal{C}$*, every morphism in* $Hom(F(A), F(B))$ *is* F *of some morphism in* $Hom(A, B)$ *(onto). A fully faithful functor is full and faithful.*

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[1] A function $f : A \longrightarrow B$, A, B being sets, is injective if $f(p) = f(q) \Longrightarrow p = q$, $p, q \in A$.

[2] A function $f : A \longrightarrow B$, being A, B sets, is surjective if $\forall b \in B$, $\exists a \in A$ s.t. $f(a) = b$.

3 Morphisms and Nash Equilibrium

Definition 8 (Strategic Game) $\Gamma = \langle N, (A_i), (\succeq_i) \rangle$ *is called strategic game, where*

(i) N *is the finite set of players,*

(ii) A_i *the nonempty set of actions available to* $i \in N$*,*

(iii) $\succeq_i$ *is the complete, transitive and reflexive preference relation on* $A = \times_{j \in N} A_j$ *of player* $i \in N$*.*

Note that the finite set of players is just an index set for actions and preferences. Therefore we may as well consider the strategic game in the reduced form:

Definition 9 (Strategic Game in Reduced Form) $G = \langle A, (\succeq_i) \rangle$ *is called strategic game in reduced form, where*

(i) $A = \times_{j \in N} A_j$ *the nonempty set of outcomes*

(ii) $\succeq_i$ *is the complete, transitive and reflexive preference relation on* A *of each player* $i \in N$*.*

Following representation result is provided:

Proposition 1 (Strategic Game in Categorical Form) *G forms a category $\mathcal{G}$ defined as follows:*

(i) $A = \mathcal{G}_0$*,*

(ii) $a, b \in A$*,* $b \succeq_i a$ *iff there is a morphism* $a \longrightarrow_i b$*,*

(iii) reflexivity of $\succeq_i$ *is forced by identity morphisms* $a \longrightarrow_i a$*,* $\forall a \in A$*,* $\forall i \in N$*,*

(iv) transitivity of $\succeq_i$ *is forced by composition of morphisms, i.e.* $a, b, c \in A$*,* $a \longrightarrow_i b$*,* $b \longrightarrow_i c$ *then by* $(a \longrightarrow_i b) \circ (b \longrightarrow_i c)$ *we have* $a \longrightarrow_i c$*,*

(v) completeness of $\succeq_i$ *requires that* $\forall a, b \in A$*, either* $a \longrightarrow_i b$ *or* $b \longrightarrow_i a$*,* $\forall i \in N$*.*

Proof. It is straightforward to verify that $\mathcal{G}$ is indeed a category. **q.e.d.**

Remark 1 *Suppose $a \longrightarrow_i b$ and $b \longrightarrow_j c$ exists. Then composition is forced by the axioms of category. Such compositions are used for example to solve for Nash equilibria in 2x2 games.*

Remark 2 *Since N is finite, $Hom_{\mathcal{G}}(a,b)$ is a finite set for all $a, b \in A$ (since for every $i \in N$ there can be at most one morphism).*

Definition 10 (Subcategory of Strategic Game) *Let $\mathcal{G}'$ be a subcategory of $\mathcal{G}$ such that*

(i) $\mathcal{G}_0 = \mathcal{G}'_0$,

(ii) $\mathcal{G}_1$ *is defined as follows:* $\forall a, b \in A$ *define sets* $Hom_{\mathcal{G}'}(a,b) \subseteq Hom_{\mathcal{G}}(a,b)$ *s.t. for* $a = (a_i, a_{-i})$ *and* $b = (b_i, a_{-i})$, $b_i \in A_i$ *for some* $i \in N$, *then*

$$Hom_{\mathcal{G}'}(a,b) = \begin{cases} \{a \longrightarrow_i b\} & \textit{if } a \longrightarrow_i b \textit{ exists,} \\ \emptyset & \textit{otherwise.} \end{cases}$$

Clearly, $\mathcal{G}'$ is a subcategory of $\mathcal{G}$. The subcategory is formed by neglecting an arrow of a player iff this player would need to make some co-operative effort to reach the co-domain of the arrow. Only arrows between outcomes the player can unilaterally reach are considered.

As a remark, there exists a full but not faithful functor $F : \mathcal{G} \longrightarrow \mathcal{G}'$. Set $F(a) = id_a(a) = a$, $\forall a \in A$,

$$F(\longrightarrow_i) = \begin{cases} \longrightarrow_i & \text{if } a_{-i} \in cod(\longrightarrow_i) \text{ and } a_{-i} \in dom(\longrightarrow_i) \\ id_{dom(\longrightarrow_i)} & \text{otherwise,} \end{cases}$$

and identity and composition preserved. This functor is full since every morphisms in $\mathcal{G}'$ is F of some morphism in $\mathcal{G}$. However, this functor is not faithful since some non-identity morhpsims in $\mathcal{G}$ are taken to identities in $\mathcal{G}'$. This functor I won't use for anything. However, it has a nice interpretation: In undergraduate game theory we do not consider a player's arrow between two outcomes if this player can not unilaterally change one outcome to the other by her actions. The functor can be interpreted as neglecting those arrows. Note that an arrow of a player is neglected iff this player would need to make some co-ordinate effort with some other player(s) to reach the codomain of the arrow. Indeed, for finding Nash equilibria in pure strategies (see below) one does not need complete preferences of the players. Rather one need the player's preference over outcomes that are affected by her.

Definition 11 (Nash Equilibrium) *A Nash equilibrium in pure strategies of an strategic game G is a profile $a^* \in A$ of actions such that $\forall i \in N$,*

$$(a_i^*, a_{-i}^*) \succeq_i (a_i, a_{-i}^*), \forall a_i \in A_i. \tag{1}$$

Denote the set of Nash equilibria of the objective strategic game by $E(G)$.

Unless otherwise note, I refer in this paper to Nash equilibrium in pure strategies simply as Nash equilibrium.

Theorem 1 *$a \in A$ is a Nash equilibrium of G iff it is the domain only of isomorphisms in $\mathcal{G}'$.*

Proof. "$\Longrightarrow$": Suppose $a \in E(G)$ but a domain of morphism f in $\mathcal{G}'$ which is not an isomophism. Then $\exists b \neq a$, $a, b \in A$ and $i \in N$ s.t. $(a_i, a_{-i}) \longrightarrow_i (b_i, a_{-i})$ in $\mathcal{G}'$ for which there is no $(b_i, a_{-i}) \longrightarrow (a_i, a_{-i})$ (otherwise it would be an isomorphism). By definition of b it follows that $a \longrightarrow_i b$ also in $\mathcal{G}$. Hence $(b_i, a_{-i}) \succeq_i (a_i, a_{-i})$ which implies $a \notin E(G)$, a contradiction.

"$\Longleftarrow$": Suppose $a \in A$ is the domain only of isomorphisms but not a Nash equilibrium. Not being a Nash equilibrium means that there $\exists b \in A$ s.t. for some $i \in N$ $b \succeq_i a$ and not $a \succeq_i b \Longleftrightarrow (b_i, a_{-i}) \succeq_i (a_i, a_{-i})$ and not $(a_i, a_{-i}) \succeq_i (b_i, a_{-i})$ in $G \Longleftrightarrow (a_i, a_{-i}) \longrightarrow_i (b_i, a_{-i})$ and not $(b_i, a_{-i}) \longrightarrow_i (a_i, a_{-i})$ in $\mathcal{G} \Longrightarrow$ (by definition of b also "$\Longleftarrow$") $a \longrightarrow_i b$ and not $b \longrightarrow_i a$ in $\mathcal{G}'$. Thus a is the domain of $a \longrightarrow_i b$ but not the codomain of an morphism $b \longrightarrow_i a$ in $\mathcal{G}'$. Hence, a is the domain of a morphism which is not an isomorphisms, a contradiction. **q.e.d**

Note that any outcome is the domain of at least one isomorphism since the identity morphism is an isomorphism.

Corollary 1 *If a terminal object in $\mathcal{G}'$ exists then it is a Nash equilibrium in G.*

Proof. Suppose there exists a terminal object $a \in A$. If for some $i \in N$, $a \in A$ is the domain of an morphism $a \longrightarrow_i b$ in $\mathcal{G}'$, then there exists by the definition of terminal object also a morphism $b \longrightarrow_i a$. Hence, every terminal object is only in the domain of isomorphisms. **q.e.d.**

Remark 3 *If $a \in A$ is terminal object in $\mathcal{G}'$ and in the domain of a morphism $a \longrightarrow_i b$ in $\mathcal{G}'$ then b is also a terminal object and a Nash equilibrium.*

Theorem 2 *Suppose $a \in A$ is a terminal object in $\mathcal{G}'$. If $\succeq_i$ is strict, then $a \in A$ is the unique Nash equilibrium in G.*

Proof. Suppose there exists a terminal object $a \in A$ which not the unique Nash equilibium, then there exists another terminal object $b \neq a$ such that for some $i \in N$, $a \longrightarrow_i b$ in $\mathcal{G}'$. Since $\succeq_i$ is strict, there is no morphism $b \longrightarrow_i a$. Hence $a \in A$ is not terminal in $\mathcal{G}'$, a contradiction. **q.e.d.**

Define $A(a) := \{b \in A : \exists i \in N, b_i \neq a_i, b_{-i} = a_{-i}\}$. That is $A(a)$ is the set of combination of actions which differ exactly in one co-ordinate from $a \in A$. Define further $Hom_{\mathcal{G}'}(A(a), a) = \cup_{b \in A(a)} Hom_{\mathcal{G}'}(b, a)$. This is the set of arrows between $A(a)$ and a.

Definition 12 $P(a) := \sharp Hom_{\mathcal{G}'}(A(a), a)$.

Theorem 3 *If $E(G) \neq \emptyset$ then $E(G) = \{a' \in A : a' \in \arg\max_{a \in A} P(a)\}$.*

Proof. Suppose $E(G) \neq \emptyset$. By definition of $E(G)$ we have each $a^* \in E(G)$, $\forall i \in N$, $(a_i, a^*_{-i}) \longrightarrow_i (a^*_i, a^*_{-i})$, $\forall a_i \in A_i$. Hence $\sharp A(a^*) = \Sigma_{i \in N}(\sharp A_i - 1) = \max_{a \in A} P(a) = P(a^*)$, $\forall a^* \in E(G)$. What is left to show is that there does not exist any $a \notin E(G)$ s.t. $P(a) = \max_{a \in A} P(a)$. Suppose the contrary, then $P(a) = A(a^*)$, for some $a^* \in E(G)$. Hence $\forall i \in N$, $(a'_i, a_{-i}) \longrightarrow_i (a_i, a_{-i})$, $\forall a'_i \in A_i$, which implies $a \in E(G)$, a contradiction. **q.e.d.**

Note that in some ordinal potential potential games, P is an ordinal potential, and in some generalized ordinal potential games, P is a generalized ordinal potential, but it does not hold in general (see [2] Monderer/Shapley (1996) for potential games). To see this consider following example of ordinal potential game (left) and it's ordinal potential given by P (right) (a similar example was given in [3] Voornefeld/Norde, 1997).

$$\begin{array}{cc} (3,1) & (1,2) \\ (4,1) & (2,1) \end{array} \qquad \begin{array}{cc} 0 & 1 \\ 2 & 2 \end{array}$$

To see a counter-example consider following game below (left) that has an ordinal potential (middle) but the function P (right) is not an ordinal potential. In this case the ordinal potential function depicted in the middle is obtained by considering the homesets $Hom_{\mathcal{G}}(A, a)$ (neglecting identity morphisms).

$$\begin{array}{cc} (0,2) & (1,3) \\ (1,0) & (0,1) \end{array} \qquad \begin{array}{cc} 0 & 3 \\ 1 & 2 \end{array} \qquad \begin{array}{cc} 0 & 2 \\ 1 & 1 \end{array}$$

To see that P can be used to find generalized ordinal potential functions in some generalized ordinal potential games, consider following example by [3] Voornefeld/Norde (1997) of generalized ordinal potential game (left), that is not an ordinal potential game, and its generalized ordinal potential given by P (right).

$$\begin{array}{ccc} (0,1) & (1,2) & (0,0) \\ (1,1) & (0,0) & (0,0) \\ (0,0) & (0,0) & (1,1) \end{array} \qquad \begin{array}{ccc} 2 & 4 & 1 \\ 4 & 2 & 2 \\ 2 & 2 & 4 \end{array}$$

References

[1] Mac Lane, Saunders (1971). *Categories for the working mathematician*, New York, Heidelberg, Berlin: Springer-Verlag.

[2] Monderer, Dov and Lloyd S. Shapley (1996). Potential games, *Games and Economic Behavior* **14**, 124-143.

[3] Voorneveld, Mark and Henk Norde (1997). A characterization of ordinal potential games, *Games and Economic Behavior* **19**, 235-242.