When is a two-person game *not really* a two-person game?*

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We can define a general n-person game as a function

$$f : S_1 \otimes S_2 \otimes S_3 \otimes \cdots \otimes S_n \longrightarrow R^n$$

where the S_i are often called the *strategy* sets (in most cases all S_i are identical). Suppose we write $f(x) = v$, where $x_j \in S_j$. Further we denote the elements of the vector v (commonly called the payoff vector) by

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} g_1(x_1, x_2, \ldots, x_n) \\ g_2(x_1, x_2, \ldots, x_n) \\ g_3(x_1, x_2, \ldots, x_n) \\ \vdots \\ g_n(x_1, x_2, \ldots, x_n) \end{pmatrix}$$

If there exists a p such that all the functions $g_1, g_2, \ldots g_p$ *only* depend on $x_1, x_2, \ldots x_p$, and further, all the functions $g_{p+1}, g_{p+2}, \ldots g_n$ *only* depend on $x_{p+1}, x_{p+2}, \ldots x_n$, then somehow the n-person game is **factorizable** (or separable?) into two games, a p-person game and an $(n-p)$-person game.

1 Example A: Prisoners Dilemma

$n = 2$ and $S_1 = S_2 = \{C, D\}$.

$$f(C, C) = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \quad f(C, D) = \begin{pmatrix} 0 \\ 5 \end{pmatrix} \quad f(D, C) = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \quad f(D, D) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

This game is not factorizable/separable.

2 Example B: A factorizable/separable game

$n = 2$ and $S_1 = S_2 = \{C, D\}$.

$$f(C, C) = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad f(C, D) = \begin{pmatrix} 4 \\ 7 \end{pmatrix} \quad f(D, C) = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \quad f(D, D) = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$$

This is factorizable/separable: Instead of a 2-person game, it is two 1-person games. In particular $g_1(x_1, x_2) \equiv g_1(x_1)$ depends only on x_1 with $g_1(C) = 4,\ g_1(D) = 5$. Also $g_2(x_1, x_2) \equiv g_2(x_2)$ depends only on x_2 with $g_2(C) = 3,\ g_2(D) = 7$.

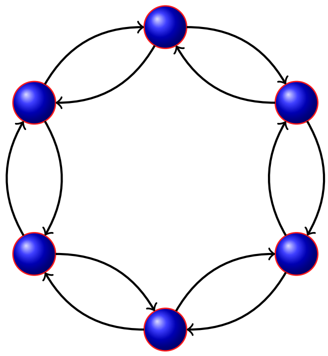
3 (Di)Graph associated with an n-person game

This graph will have n nodes. We draw a directed arc **from** node i **to** node j if and only if g_i actually depends on the choice of x_j. We observe that the n-person game is factorizable/separable if and only if its associated graph is not connected. Are there other properties of the graph (existence of cycles, etc.) from which one can deduce results about the associated game? Has this been studied?

references welcome

3.1 Example C: Six players playing Prisoners Dilemma in a ring

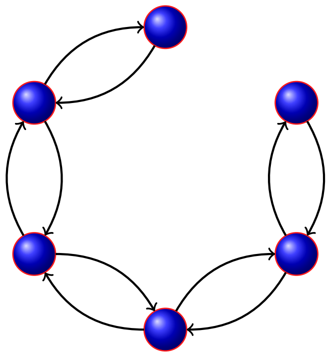
The payoffs are given in the following table (in each case, we average over the number of games played by each player):



x_{i-1}	C	C	C	C	D	D	D	D
x_i	C	C	D	D	C	C	D	D
x_{i+1}	C	D	C	D	C	D	C	D
$g_i(x_1, x_2, \dots x_6)$	3	1.5	5	3	1.5	0	3	1

3.2 Example D: Six players playing Prisoners Dilemma in a line

Effectively this corresponds to breaking one of the links in the ring. The payoffs are given in the following table (in each case, we average over the number of games played by each player):



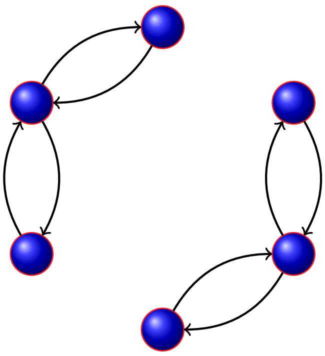
x_{i-1}	C	C	C	C	D	D	D	D
x_i	C	C	D	D	C	C	D	D
x_{i+1}	C	D	C	D	C	D	C	D
For $1 < i < 6 : g_i(x_1, x_2, \dots x_6)$	3	1.5	5	3	1.5	0	3	1
For $i = 1 : g_i(x_1, x_2, \dots x_6)$	3	0	5	1	3	0	5	1
For $i = 6 : g_i(x_1, x_2, \dots x_6)$	3	3	5	5	0	0	1	1

3.3 Example E: Two groups of three players

This corresponds to breaking two of the links in the ring. The payoffs are given in the following table (in each case, we average over the number of games played by each player):

x_{i-1}	C	C	C	C	D	D	D	D
x_i	C	C	D	D	C	C	D	D
x_{i+1}	C	D	C	D	C	D	C	D
For $i = 2, 5 : g_i(x_1, x_2, \dots x_6)$	3	1.5	5	3	1.5	0	3	1
For $i = 1, 4 : g_i(x_1, x_2, \dots x_6)$	3	0	5	1	3	0	5	1
For $i = 3, 6 : g_i(x_1, x_2, \dots x_6)$	3	3	5	5	0	0	1	1

*Short answer: When it is really two one-person games.



4 Relevance to quantum games played on a network

In quantum games, one has the (non-classical) aspect of **entanglement**. This creates “links” between players. We’d like to know - can/does entanglement make a separable/factorizable game into a connected one? Further, for non - separable/factorizable games, does entanglement change the structure of the associated graph?